# ANALYSIS OF A TAPERED CIRCULAR WAVEGUIDE USING SPHERICAL MODES

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## Abstract

In this paper the coupled mode equations governing the propagation of $TE^{\circ}_{0m}$ waves in a tapered circular waveguide are derived. The derivation is based on the assumption that an elementary section of the taper can be thought of as an elementary truncated cone. This approach is superior to the cylindrical mode representation in that it eliminates the singularity in the coupled mode equations for the modal amplitudes at "cut-off", and also leads to a faster converging expansion. An iterative method of solving these equations is also outlined.

## Introduction

In the course of developing a millimeter waveguide transmission system, transitions between two circular cylindrical waveguides of different radii are used frequently. These tapered sections are designed so as to transmit the $TE^{\circ}_{01}$ mode freely, at the same time keeping the excited $TE^{\circ}_{02}$ mode level as low as possible. The formulae used to design the tapers at the present time are based on an approximation which treats an elementary section of the taper as an elementary section of a uniform cylindrical waveguide. Under this formulation the $TE^{\circ}_{02}$ mode could have a cut-off point, under certain conditions, somewhere inside the taper which introduces difficulties, in the form of singularities, in the analysis and design of these components. Even though in practice this difficulty is bypassed by designing the tapers at high enough frequencies, so that both modes propagate at all sections, it is desirable to be able to analyze them at lower frequencies. The spherical mode representation has the advantage that the transition between the propagating and the evanescent regimes of the $TE^{\circ}_{02}$ mode is made in a continuous manner and therefore can be used to investigate the performance of tapers at all frequencies.

Mode conversion in a tapered waveguide capable of multimode propagation has been investigated by a number of authors. The method employed is to reduce the boundary value problem to a set of coupled differential equations (telegraphist's equations). This is accomplished by expressing the electromagnetic field at any cross section in terms of an infinite set of cylindrical basis functions characteristic of a uniform waveguide of the same cross section[1-3]. These basis functions, do not, in general satisfy the boundary condition individually and hence the infinite series representing the total field converges very slowly, or may even fail to converge. In practice one must truncate the infinite series after two or three terms and so the slow convergence becomes a real problem. This problem has been recognized in the past and efforts have been made to overcome this difficulty. Unger[4] formulates the problem in a "natural" coordinate system but this method fails when the curvature of the taper profile becomes negative. A satisfactory method of improving the convergence of the telegraphist's equation has been formulated by Bahar[5,6]. In his analysis the nonuniform waveguide is considered to consist of an infinite number of elementary sections. In each of these elementary sections the field is expressed in terms of an infinite complete set of "local" basis functions that individually satisfy Maxwell's equations and the exact boundary conditions. The differential scattering coefficients for the junction between two adjacent elementary sections are obtained by solving the boundary value problem and in the limit as the size of these sections goes to zero these coefficients lead to the telegraphist's equations.

Consider a tapered section of length L, which connects two uniform circular waveguides of radii $a_1$ and $a_2$ ($a_1 < a_2$) (FIG. 1). A $TE^{\circ}_{0m}$ wave propagating in the waveguide of radius $a_1$ is incident on the taper waveguide combination. It is desired to compute the levels of the various $TE^{\circ}_{0n}$ modes excited in the two waveguides. Assume the tapered section to be made up of a very large number of elementary conical sections. Let A, B denote two adjacent elementary conical sections at z and z + dz, respectively, (FIG. 2). The fields in A and B can be represented in terms of spherical modes with origins at z' and z' + dz', respectively. Specifically, the local $TE^{\circ}_{0m}$ field components can be written in terms of a scalar potential M such that

$$M \sim h_{\nu}^{(j)}(kr)\, P_{\nu}(\cos\theta) \quad . \qquad (1)$$

where $h_{\nu}^{(j)}(kr) = (\pi/2\,kr)^{1/2}\, H_{\nu+1/2}^{(j)}(kr)$, $H_{\nu}$ denoting the Hankel function, $\nu$ takes on a discrete set of values $\{\nu_p\}$, p = 1,2,3,... which are the roots of $P_{\nu}^{1}(\cos\psi) = 0$, $\psi$

being the local cone angle and r and $\theta$ are the spherical coordinates with origins at the tips of the local conical sections. Let $a_m^A$, $a_m^B$ denote the amplitudes of the forward propagating $TE^{\circ}_{0m}$ waves (in the + z sense) in sections A and B, respectively, and $b_m^A$, $b_m^B$ denote the amplitudes of the corresponding backward waves (FIG. 3). The differential scattering coefficients $ds_{nm}^{BA}$, $ds_{nm}^{BB}$, $ds_{nm}^{AB}$ and $ds_{nm}^{AA}$ are defined by the following equations.

$$a_n^B = \sum_{m=1}^{\infty} (\delta_{nm} + ds_{nm}^{BA})\, a_m^A + \sum_{m=1}^{\infty} ds_{nm}^{BB}\, b_m^B \quad , \tag{2}$$

$$b_n^A = \sum_{m=1}^{\infty} (\delta_{nm} + ds_{nm}^{AB})\, b_m^B + \sum_{m=1}^{\infty} ds_{nm}^{AA}\, a_m^A \quad . \tag{3}$$

The next step is to match the tangential fields in Sections A and B. This is done on the surface $S^B$ (FIG. 3) and requires an analytic continuation of the fields in section A. It is felt that this procedure is justified since in the limit as $dz \to 0$ $S^A$ and $S^B$ coincide. Matching of $\bar{E}_t$ and $\bar{H}_t$ leads to explicit expressions for the differential scattering coefficients in terms of the geometry of the tapered section. To convert equations (2) and (3) into a set of coupled differential equations with z as the independent variable we proceed as follows.

Consider the profile of the taper as shown in FIG. 1. For a given z we can draw a line perpendicular to the z axis which will intersect the profile at some point P. The tangent to the profile at P will intersect the axis at some point z'. If we imagine a spherical coordinate system with origin at z' then we defineea $a_n(z)$ and $b_n(z)$ as the amplitudes of the forward and backward $TE_{0n}$ waves, respectively, on the spherical surface passing through P and centered at z'. According to this definition $a_n^A$, $a_n^B$, $b_n^A$ and $b_n^B$ are related to $a_n(z)$ and $b_n(z)$ by the relations

$$a_n^B - a_n^A = \left[\frac{da_n}{dz} + j(\beta_n^+/\cos\psi)a_n\right]dz, \tag{4}$$

$$b_n^B - b_n^A = \left[\frac{db_n}{dz} - j(\beta_n^-/\cos\psi)b_n\right]dz. \tag{5}$$

Substitution of equations (2), (3) into (3), (4) leads to the following coupled mode equations.

$$\frac{da_n}{dz} + j\psi_n^+ = f_n \quad , \tag{6}$$

$$\frac{db_n}{dz} - j\psi_n^- = g_n \quad . \tag{7}$$

where

$$\psi_n^+ = (\beta_n^+/\cos\psi) + j\,\frac{ds_{nn}^{BA}}{dz} \quad , \tag{8}$$

$$\psi_n^- = (\beta_n^-/\cos\psi) + j\,\frac{ds_{nn}^{AB}}{dz} \quad , \tag{9}$$

$$f_n(z) = \sum_{m=1}^{\infty}{}' \frac{ds_{nm}^{BA}}{dz}\, a_m + \sum_{m=1}^{\infty} \frac{ds_{nm}^{BB}}{dz}\, b_m \quad , \tag{10}$$

$$g_n(z) = -\sum_{m=1}^{\infty} \frac{ds_{nm}^{AA}}{dz}\, a_m - \sum_{m=1}^{\infty}{}' \frac{ds_{nm}^{AB}}{dz}\, b_m \quad . \tag{11}$$

and $\sum_{m=1}^{\infty}{}'$ means that the m=n term is excluded from the summation.

The coupled mode equations (6), (7) along with the boundary conditions at z=0 and z=L lead to the following integral equations.

$$a_n(z) = \exp\left\{-j\int_0^z \psi_n^+(x)\,dx\right\}\left[\int_0^z f_n(x) \cdot \exp\left\{j\int_0^x \psi_n^+(x')\,dx'\right\} dx + a_n(0)\right], \tag{12}$$

$$b_n(z) = \exp\left\{j\int_L^z \psi_n^-(x)\,dx\right\}\left[\int_L^z g_n(x) \cdot \exp\left\{-j\int_L^x \psi_n^-(x')\,dx'\right\} dx + b_n(L)\right]. \tag{13}$$

For a well designed taper the amplitudes $b_n$ are negligible in comparison with the amplitudes $a_n$. Under this approximation it is feasible to solve Equations (12), (13) by the method of successive approximation. For example, if a $TE_{01}$ mode of amplitude $a_1(0)$ is incident on the taper at z=0 we obtain in the zeroth order approximation

$$a_1^{(0)}(z) = a_1(0) \exp\left\{-j\int_0^z \psi_1^+(x)\,dx\right\}, \qquad (14)$$

all the other amplitudes being zero. Using Equation (14) to compute first order expressions $f_n^{(1)}(z)$ and $g_n^{(1)}(z)$ via Equations (10), (11) we obtain in the first order

$$a_n^{(1)}(z) = \exp\left\{-j\int_0^z \psi_n^+(x)dx\right\}\left[\int_0^z f_n^{(1)}(x) \exp\left\{j\int_0^x \psi_n^+(x')dx'\right\}dx + a_1(0)\,\delta_{n1}\right], \qquad (15)$$

$$b_n^{(1)}(z) = \exp\left\{j\int_0^z \psi_n^-(x)dx\right\}\left[\int_L^z g_n^{(1)}(x) \exp\left\{-j\int_L^x \psi_n^-(x')dx'\right\}dx\right]. \qquad (16)$$

This process may be continued until convergence is obtained.

To compute the differential scattering coefficients numerically, it is necessary to compute Legendre functions of order $\nu_m$

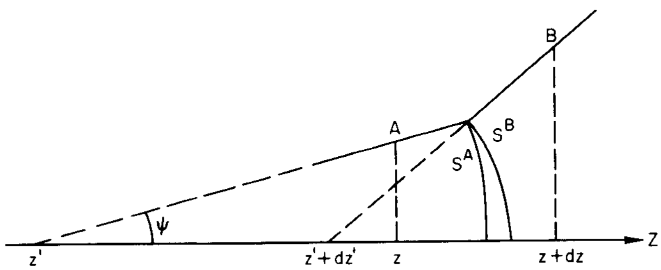


FIG. 2. Adjacent Elementary Conical Sections

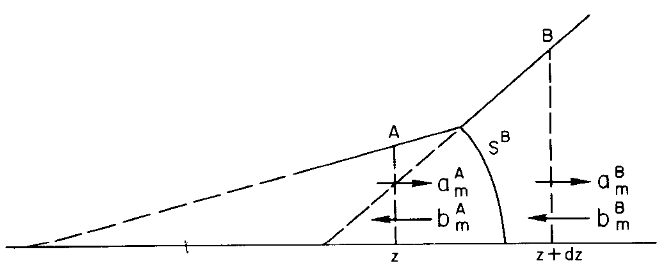


FIG. 3. Forward and Backward Waves

and Hankel functions of order $\nu_m + 1/2$. It can be shown that when the local cone angle $\psi$ is small $\nu_m = (k_m/\psi)$ approximately where $k_m$ denote the roots of the Bessel function $J_1$. Thus, we see that $\nu_m$ is very large for very small local cone angles. The Hankel functions of large orders and arguments can be computed from their uniform asymptotic expansions and the Legendre function by numerical quadrature.

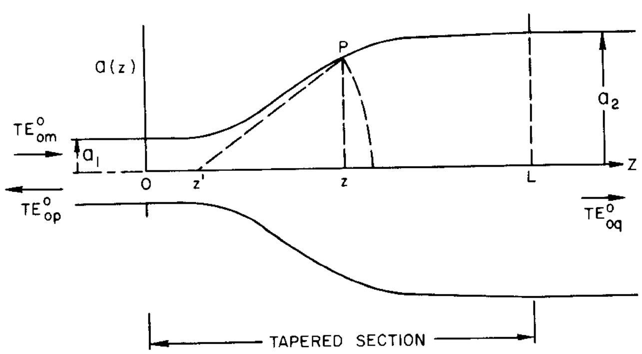


FIG. 1. Tapered Section Joining Two Uniform Waveguides

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